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ORIGINAL PAPER

A two-dimensional Haar wavelet approach for the numerical simulations of time and space fractional Fokker–Planck equations in modelling of anomalous diffusion systems

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Abstract In this paper, the numerical solution for the fractional order Fokker–Planck equation has been presented using two dimensional Haar wavelet collocation method. Two dimensional Haar wavelet method is applied to compute the numerical solution of nonlinear time- and space-fractional Fokker–Planck equation. The approximate solutions of the nonlinear time- and space-fractional Fokker–Planck equation are compared with the exact solutions as well as solutions available in open literature. The present scheme is very simple, effective and convenient for obtaining numerical solution of the time and space-fractional Fokker–Planck equation.

Keywords Fractional Fokker–Planck equation · Haar wavelet method · Caputo derivative

1 Introduction

Fractional calculus is a field of applied mathematics which deals with derivatives and integrals of arbitrary orders. The fractional calculus has gained considerable importance during the past decades mainly due to its application in diverse fields of science and engineering such as viscoelasticity, diffusion of biological population, signal processing, electromagnetism, fluid mechanics, electrochemistry and many more. Fractional differential equations are extensively used in modeling of physical phenomena in various fields of science and engineering. For this we need a reliable and efficient technique for the solution of fractional differential equations.

Recently, orthogonal wavelets bases are becoming more popular for numerical solutions of partial differential equations due to their excellent properties such as ability

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to detect singularities, orthogonality, flexibility to represent a function at different level of resolution and compact support. In recent years, there has been a growing interest in developing wavelet based numerical algorithms for solution of fractional order partial differential equations. Among them, the haar wavelet method is the simplest and is easy to use. Haar wavelets have been successfully applied to the solutions of ordinary and partial differential equations, integral equations and integro-differential equations. Therefore, the main focus of the present paper is the application of Haar wavelet technique to solve the problem of time and space fractional Fokker–Planck equations. The obtained numerical approximate results of this method are then compared with the exact solutions as well as solutions available in open literature.

Fokker–Planck equation (FPE) was introduced by Adriaan Fokker and Max Planck, commonly used to describe the Brownian motion of particles [1]. A FPE describes the change of probability of a random function in space and time; hence it is naturally used to describe solute transport. The general FPE for the motion of a concentration field $u(x,t)$ of one space variable x at time t has the form [2–4]

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x} A(x) + \frac{\partial^2}{\partial x^2} B(x) \right] u(x,t), \tag{1.1}$$

with the initial condition

$$u(x,0) = f(x), x \in \Re \tag{1.2}$$

where $A(x)$ and $B(x) > 0$ are referred as the drift and diffusion coefficients. The drift and diffusion coefficients may also depend on time.

There is a more general form of FPE called nonlinear Fokker–Planck equation which is of the form [2–4]

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x} A(x,t,u) + \frac{\partial^2}{\partial x^2} B(x,t,u) \right] u(x,t), \tag{1.3}$$

The nonlinear Fokker–Planck equation (FPE) has important applications in various fields such as plasma physics, surface physics, population dynamics, biophysics, engineering, neuroscience, nonlinear hydrodynamics, polymer physics, laser physics, pattern formation, psychology and marketing etc. [5].

In recent years there has been a great deal of interest in fractional diffusion equations. These equations arise in continuous time random walks, modelling of anomalous diffusive and subdiffusive systems, unification of diffusion and wave propagation phenomenon etc. [6].

Consider the generalized nonlinear time- and space-fractional Fokker–Planck equation [7]

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \left[-\frac{\partial^\beta}{\partial x^\beta} A(x,t,u) + \frac{\partial^{2\beta}}{\partial x^{2\beta}} B(x,t,u) \right] u(x,t), t > 0, x > 0 \tag{1.4}$$

where α and β are parameters describing the order of the fractional time and space derivatives, respectively. The function $u(x, t)$ is assumed to be a casual function of time and space, i.e. vanishing for $t < 0$ and $x < 0$. The fractional derivatives are considered in the Caputo sense.

Various mathematical methods such as the Adomian decomposition method (ADM) [8], variational iteration method (VIM) [8], operational Tau method (OTM) [9] and homotopy perturbation method (HPM) [10] have been used in attempting to solve fractional Fokker–Planck equations. Our aim in the present work is to implement two dimensional Haar wavelet method in order to demonstrate the capability of this method in handling nonlinear equations of arbitrary order, so that one can apply it to various types of nonlinearity.

2 Fractional derivative and integration

There are several approaches to define the derivatives of fractional order such as Grünwald–Letnikov, Riemann–Liouville and Caputo. Riemann–Liouville fractional derivative is not suitable for real world physical problems as it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations.

Definition 1 The most frequently encountered definition of an integral of fractional order is the Riemann–Liouville integral in which the fractional integral operator J^{α} $(\alpha > 0)$ of a function $f(t)$, is defined as [11,12]

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-\tau)^{\alpha-1} f(\tau) \quad d\tau, \alpha > 0 \text{ and } \alpha \in \Re^{+} \tag{2.1}$$

where $\Gamma(.)$ is the well-known gamma function, and some properties of the operator J^{α} are as follows

$$J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t), \quad (\alpha > 0, \beta > 0) \tag{2.2}$$

$$J^{\alpha} t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)} t^{\alpha+\gamma}, \quad (\gamma > -1) \tag{2.3}$$

Definition 2 The fractional derivative introduced by Caputo [11,12], in the late sixties, is called Caputo Fractional derivative. The Caputo fractional derivative ${}_0D_t^{\alpha}$ of a function $f(t)$ is defined as [11,12]

$${}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} \frac{f^{n}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (n-1 < \alpha \leq n, n \in N) \tag{2.4}$$

The following are two basic properties of the Caputo fractional derivative

$$ {}_0D_t^{\alpha} t^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}, \quad 0<\alpha<\beta+1, \beta>-1 \tag{2.5} $$

$$ J^{\alpha} D^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}\left(0^{+}\right) \frac{t^k}{k!}, \quad n-1<\alpha\le n \text{ and } n\in N \tag{2.6} $$

3 Haar wavelets

Haar functions have been used from 1910 when they were introduced by the Hungarian mathematician Alfred Haar. Haar wavelets are the simplest wavelets among various types of wavelets. They are step functions over the real line and can take only three values 0, 1 and -1. The method has been used for being its simpler, fast and computationally attractive feature. Usually the Haar wavelets are defined for the interval $t \in [0,\ 1)$ but in general case $t \in [A,\ B]$, we divide the interval $[A, B]$ into m equal subintervals; each of width $\Delta t = (B-A)/m$. In this case, the orthogonal set of Haar functions are defined in the interval $[A, B]$ by [13]

$$ h_0(t) = \begin{cases} 1 & t\in\ [A, B], \\ 0 & elsewhere, \end{cases} \tag{3.1} $$

$$ \text{and } h_i(t) = \begin{cases} 1, & \zeta_1(i) \le t < \zeta_2(i) \\ -1, & \zeta_2(i)\le t < \zeta_3(i) \\ 0, & otherwise \end{cases} \tag{3.2} $$

where

$$ \zeta_1(i) = A + \left(\frac{k-1}{2^j}\right)(B-A) = A + \left(\frac{k-1}{2^j}\right) m\Delta t, $$
$$ \zeta_2(i) = A + \left(\frac{k-(1/2)}{2^j}\right)(B-A) = A + \left(\frac{k-(1/2)}{2^j}\right) m\Delta t, $$
$$ \zeta_3(i) = A + \left(\frac{k}{2^j}\right)(B-A) = A + \left(\frac{k}{2^j}\right) m\Delta t, $$

$i = 1, 2, \ldots, m, \quad m = 2^J$ and J is a positive integer which is called the maximum level of resolution. Here j and k represent the integer decomposition of the index i. i.e. $i = k + 2^j - 1, 0 \le j < i$ and $1 \le k < 2^j + 1$.

4 Function approximation

Any function $y(t) \in L^2([0,\ 1))$ can be expanded into Haar wavelets by [13–15]

$$ y(t) = c_0h_0(t) + c_1h_1(t) + c_2h_2(t) + \cdots, \quad c_j = \int_0^1 y(t)h_j(t)dt. \tag{4.1} $$

If $y(t)$ is approximated as piecewise constant in each subinterval, the sum in Eq. (4.1) may be terminated after m terms and consequently we can write discrete version in the matrix form as

$$\mathbf{y} \approx \sum_{i=0}^{m-1} c_i h_i(t_l) = \quad \boldsymbol{C}_{\boldsymbol{m}}^{\boldsymbol{T}} H_m, \tag{4.2}$$

where $\boldsymbol{Y}$ and $\boldsymbol{C}_m^T$ are m-dimensional row vectors.

Here H is the Haar wavelet matrix of order m defined by $H = [\boldsymbol{h}_0, \boldsymbol{h}_1, \ldots, \boldsymbol{h}_{m-1}]^T$ i.e.

$$H = \begin{bmatrix} \boldsymbol{h}_0 \\ \boldsymbol{h}_1 \\ \cdots \\ \boldsymbol{h}_{m-1} \end{bmatrix} = \begin{bmatrix} h_{0,0} & h_{0,1} & \cdots & h_{0,m-1} \\ h_{1,0} & h_{1,1} & \cdots & h_{1,m-1} \\ \cdots & & & \\ h_{m-1,0} & h_{m-1,1} & \cdots & h_{m-1,m-1} \end{bmatrix}, \tag{4.3}$$

where $\boldsymbol{h}_0, \boldsymbol{h}_1, \ldots, \boldsymbol{h}_{m-1}$ are the discrete form of the Haar wavelet bases.

The collocation points are given by

$$t_l = A + (l - 0.5)\,\Delta t_, \quad l = 1, 2, \ldots, m \tag{4.4}$$

5 Operational matrix of the general order integration

The integration of the $H_m(t) = \left[h_0(t),\ h_1(t), \ldots, h_{m-1}(t)\right]^T$ can be approximated by [15]

$$\int_0^t H_m(\tau)d\tau \cong QH_m(t), \tag{5.1}$$

where Q is called the Haar wavelet operational matrix of integration which is a square matrix of m-dimension. To derive the Haar wavelet operational matrix of the general order of integration, we recall the fractional integral of order $\alpha(>0)$ which is defined by Podlubny [11]

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad \alpha > 0, \quad \alpha \in \Re^+ \tag{5.2}$$

where $\Re^+$ is the set of positive real numbers.

The Haar wavelet operational matrix Q^{α} for integration of the general order α is given by

$$\begin{aligned} Q^{\alpha} H_m(t) = J^{\alpha} H_m(t) &= \left[J^{\alpha} h_0(t), J^{\alpha} h_1(t), \ldots, J^{\alpha} h_{m-1}(t)\right]^T \\ &= \left[Q h_0(t), Q h_1(t), \ldots, Q h_{m-1}(t)\right]^T \end{aligned} \tag{5.3}$$

where

$$Qh_0(t) = \begin{cases} \frac{t^{\alpha}}{\Gamma(1+\alpha)}, & t \in \ [A,\ B]\ , \\ 0, & elsewhere\ , \end{cases} \tag{5.4}$$

and

$$Qh_i(t) = \begin{cases} 0, & A \le t < \zeta_1(i)\ , \\ \phi_1, & \zeta_1(i) \le t < \zeta_2(i), \\ \phi_2, & \zeta_2(i) \le t < \zeta_3(i), \\ \phi_3, & \zeta_3(i) \le t < B, \end{cases} \tag{5.5}$$

where

$$\begin{aligned} \phi_1 &= \frac{(t-\zeta_1(i))^{\alpha}}{\Gamma(\alpha+1)}, \\ \phi_2 &= \frac{(t-\zeta_1(i))^{\alpha}}{\Gamma(\alpha+1)} - 2\frac{(t-\zeta_2(i))^{\alpha}}{\Gamma(\alpha+1)}, \\ \phi_3 &= \frac{(t-\zeta_1(i))^{\alpha}}{\Gamma(\alpha+1)} - 2\frac{(t-\zeta_2(i))^{\alpha}}{\Gamma(\alpha+1)} + \frac{(t-\zeta_3(i))^{\alpha}}{\Gamma(\alpha+1)}, \end{aligned}$$

for $i = 1, 2, \ldots, m$, $m = 2^J$ and J is a positive integer, called the maximum level of resolution. Here j and k represent the integer decomposition of the index i. i.e. $i = k + 2^j - 1,\ 0 \le j < i$ and $1 \le k < 2^j + 1$.

6 Application of two dimensional Haar wavelet for solving time fractional Fokker–Planck equation

Consider the nonlinear time-fractional Fokker–Planck equation [8,10]

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \left[-\frac{\partial}{\partial x}\left(\frac{4u}{x} - \frac{x}{3}\right) + \frac{\partial^2 u}{\partial x^2}\right] u\,(x,t)\,,\ t > 0, x > 0 \tag{6.1}$$

where $0 < \alpha \le 1$, subject to the initial condition

$$u\,(x, 0) = x^2 \tag{6.2}$$

When $\alpha = 1$,the exact solution of Eq. (6.1) is given by [8,10]

$$u(x,t) = x^2 e^t \tag{6.3}$$

Let us divide both space and time interval [0, 1] into m equal subintervals; each of width $\Delta = \frac{1}{m}$.

Haar wavelet solution of $u(x,t)$ is sought by assuming that $\frac{\partial^2 u(x,t)}{\partial x^2}$ can be expanded in terms of Haar wavelets as

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij} h_i(x) h_j(t) \tag{6.4}$$

Integrating Eq. (6.4) w.r.t. x from 0 to x we get

$$\frac{\partial u(x,t)}{\partial x} - p(t) = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij} Q h_i(x) h_j(t) \tag{6.5}$$

Again, integrating Eq. (6.5) w.r.t. x from 0 to x we get

$$u(x,t) = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij} Q^2 h_i(x) h_j(t) + q(t) + x p(t) \tag{6.6}$$

Putting $x = 0$, in Eq. (6.6) we get

$$q(t) = u(0,t) = 0 \tag{6.7}$$

Putting $x = 1$, in Eq. (6.6) we get

$$\begin{aligned} p(t) &= u(1,t) - u(0,t) - \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij} \left[Q^2 h_i(x)\right]_{x=1} h_j(t) \\ &= u(1,t) - \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij} \left[Q^2 h_i(x)\right]_{x=1} h_j(t) \end{aligned} \tag{6.8}$$

Putting Eq. (6.8) in Eqs. (6.5) and (6.6) we have

$$\frac{\partial u(x,t)}{\partial x} = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij} Q h_i(x) h_j(t) + u(1,t) - \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij} \left[Q^2 h_i(x)\right]_{x=1} h_j(t) \tag{6.9}$$

$$u(x,t) = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij} Q^2 h_i(x) h_j(t) + x\left[u(1,t) - \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij} \left[Q^2 h_i(x)\right]_{x=1} h_j(t)\right] \tag{6.10}$$

The nonlinear term presented in Eq. (6.1) can be approximated using Haar wavelet function as

$$\frac{\partial^2 u^2}{\partial x^2} - \frac{\partial}{\partial x}\left(\frac{4u^2}{x}\right) = \sum_{i=1}^{m}\sum_{j=1}^{m} d_{ij} h_i(x) h_j(t)$$

This implies

$$2u\left(\frac{\partial^2 u}{\partial x^2} + \frac{2u}{x^2}\right) + 2\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial x} - \frac{4u}{x}\right) = \sum_{i=1}^{m}\sum_{j=1}^{m} d_{ij} h_i(x) h_j(t) \tag{6.11}$$

Therefore substituting Eqs. (6.4), (6.9) and (6.10) in Eq. (6.11) we have

$$\begin{aligned}
&2\left(\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij} Q^2 h_i(x) h_j(t) + x\left[u(1,t) - \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}\left[Q^2 h_i(x)\right]_{x=1} h_j(t)\right]\right)\\
&\left[\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij} h_i(x) h_j(t) + \frac{2}{x^2}\left(\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij} Q^2 h_i(x) h_j(t)\right.\right.\\
&\quad\left.\left. + x\left[u(1,t) - \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}\left[Q^2 h_i(x)\right]_{x=1} h_j(t)\right]\right)\right]\\
&\quad + 2\left(\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij} Q h_i(x) h_j(t) + u(1,t) - \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}\left[Q^2 h_i(x)\right]_{x=1} h_j(t)\right)\\
&\left[\left(\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij} Q h_i(x) h_j(t) + u(1,t)\right.\right.\\
&\quad\left. - \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}\left[Q^2 h_i(x)\right]_{x=1} h_j(t)\right)\\
&\quad\left. - \frac{4}{x}\left(\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij} Q^2 h_i(x) h_j(t) + x\left[u(1,t) - \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}\left[Q^2 h_i(x)\right]_{x=1} h_j(t)\right]\right)\right]\\
&\quad = \sum_{i=1}^{m}\sum_{j=1}^{m} d_{ij} h_i(x) h_j(t)
\end{aligned} \tag{6.12}$$

Substituting Eq. (6.11) in Eq. (6.1) we will have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \sum_{i=1}^{m}\sum_{j=1}^{m} d_{ij} h_i(x) h_j(t) + \frac{x}{3}\frac{\partial u}{\partial x} + \frac{u}{3} \tag{6.13}$$

Now applying J^{α} to both sides of Eq. (6.13) yields

$$u(x,t) - u(x,0) = J^{\alpha}\left(\sum_{i=1}^{m}\sum_{j=1}^{m} d_{ij}h_i(x)h_j(t) + \frac{x}{3}\frac{\partial u}{\partial x} + \frac{u}{3}\right) \tag{6.14}$$

Substituting Eqs. (6.2), (6.9) and (6.10) in Eq. (6.14) we get

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}Q^2h_i(x)h_j(t) + x\left[u(1,t) - \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}\left[Q^2h_i(x)\right]_{x=1}h_j(t)\right]\\
&-x^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} d_{ij}h_i(x)Q_t^{\alpha}h_j(t)\\
&+\frac{x}{3}\left(\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}Qh_i(x)Q_t^{\alpha}h_j(t) + J^{\alpha}u(1,t)\right.\\
&\left.-\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}\left[Q^2h_i(x)\right]_{x=1}Q_t^{\alpha}h_j(t)\right)\\
&+\frac{1}{3}\left(\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}Q^2h_i(x)Q_t^{\alpha}h_j(t)\right.\\
&\left.+x\left[J^{\alpha}u(1,t) - \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}\left[Q^2h_i(x)\right]_{x=1}Q_t^{\alpha}h_j(t)\right]\right)
\end{aligned} \tag{6.15}$$

Now substituting the collocation points $x_l = \frac{l-0.5}{m}$ and $t_k = \frac{k-0.5}{m}$ for $l, k = 1, 2, \ldots, m$ in Eqs. (6.12) and (6.15), we have $2m^2$ equations in $2m^2$ unknowns in c_{ij} and d_{ij}. By solving these system of equations using mathematical software, the Haar wavelet coefficients c_{ij} and d_{ij} can be obtained.

7 Application of two dimensional Haar wavelet for solving time- and space-fractional Fokker–Planck equation

Consider the time- and space-fractional Fokker–Planck equation [8,10]

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \left[-\frac{\partial^{\beta}}{\partial x^{\beta}}\left(\frac{x}{6}\right) + \frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(\frac{x^2}{12}\right)\right]u(x,t), \quad t > 0, x > 0 \tag{7.1}$$

where $0 < \alpha, \beta \le 1$, subject to the initial condition

$$u(x,0) = x^2 \tag{7.2}$$

When $\alpha = 1$ and $\beta = 1$, the exact solution of Eq. (7.1) is given by [8,10]

$$u(x,t) = x^2 e^{\frac{t}{2}} \tag{7.3}$$

Let us divide space interval [0, 1] into m equal subintervals; each of width $\Delta = \frac{1}{m}$.

Haar wavelet solution of $u(x,t)$ is sought by assuming that $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ can be expanded in terms of Haar wavelets as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij} h_i(x) h_j(t) \tag{7.4}$$

Applying J^α both sides of Eq. (7.4), we get

$$u(x,t) = x^2 + \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij} h_i(x) Q^\alpha h_j(t) \tag{7.5}$$

Now

$$\frac{\partial^\beta}{\partial x^\beta}(xu(x,t)) = \frac{\partial^\beta}{\partial x^\beta}\left(x^3\right) + \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij} \frac{\partial^\beta}{\partial x^\beta}(xh_i(x)) Q^\alpha h_j(t) \tag{7.6}$$

and

$$\frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(x^2 u(x,t)\right) = \frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(x^4\right) + \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij} \frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(x^2 h_i(x)\right) Q^\alpha h_j(t) \tag{7.7}$$

Substituting Eqs. (7.4), (7.6) and (7.7) in Eq. (7.1) we get

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij} h_i(x) h_j(t) &= \frac{-1}{6}\left(\frac{\partial^\beta}{\partial x^\beta}\left(x^3\right) + \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}\frac{\partial^\beta}{\partial x^\beta}(xh_i(x)) Q^\alpha h_j(t)\right)\\ &\quad + \frac{1}{12}\left(\frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(x^4\right) + \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}\frac{\partial^{2\beta}}{\partial x^{2\beta}}\left(x^2 h_i(x)\right) Q^\alpha h_j(t)\right)\end{aligned} \tag{7.8}$$

Now substituting the collocation points $x_l = \frac{l-0.5}{m}$ for $l = 1, 2, \ldots, m$ in Eq. (7.8), we have m^2 equations in m^2 unknowns a_{ij}. By solving this system of equations using mathematical software, the Haar wavelet coefficients a_{ij} can be obtained.

8 Convergence analysis of two-dimensional Haar wavelet method

In this section, we have introduced the error analysis for the two-dimensional Haar wavelet method.

We assume that, $f(x, y) \in C^2([a, b] \times [a, b])$ and there exist $M > 0$; for which

$$\left|\frac{\partial^2 f(x, y)}{\partial x \partial y}\right| \leq M, \quad \forall x, y \in [a, b] \times [a, b].$$

Next, we may proceed as follows, suppose $f_{nm}(x, y) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} c_{ij} h_i(x) h_j(y)$, where, $n = 2^{\alpha+1}$, $\alpha = 0, 1, 2, \ldots$ and $m = 2^{\beta+1}$, $\beta = 0, 1, 2, \ldots$.

Then,

$$\begin{aligned} f(x, y) - f_{nm}(x, y) &= \sum_{i=n}^{\infty} \sum_{j=m}^{\infty} c_{ij} h_i(x) h_j(y) \\ &\quad + \sum_{i=n}^{\infty} \sum_{j=0}^{m-1} c_{ij} h_i(x) h_j(y) + \sum_{i=0}^{n-1} \sum_{j=m}^{\infty} c_{ij} h_i(x) h_j(y). \end{aligned}$$

From Parseval's formula, we have

$$\begin{aligned} \| f(x, y) - f_{nm}(x, y) \|^2 &= \int_a^b \int_a^b (f(x, y) - f_{nm}(x, y))^2 dx dy \\ &= \sum_{p=n}^{\infty} \sum_{s=m}^{\infty} \sum_{i=n}^{\infty} \sum_{j=m}^{\infty} c'_{ij} c'_{ps} \int_a^b h_i(x) h_p(x) dx \int_a^b h_j(y) h_s(y) dy \\ &\quad + \sum_{p=n}^{\infty} \sum_{s=0}^{m-1} \sum_{i=n}^{\infty} \sum_{j=0}^{m-1} c'_{ij} c'_{ps} \int_a^b h_i(x) h_p(x) dx \int_a^b h_j(y) h_s(y) dy \\ &\quad + \sum_{p=0}^{n-1} \sum_{s=m}^{\infty} \sum_{i=0}^{n-1} \sum_{j=m}^{\infty} c'_{ij} c'_{ps} \int_a^b h_i(x) h_p(x) dx \int_a^b h_j(y) h_s(y) dy \\ &= \sum_{i=n}^{\infty} \sum_{j=m}^{\infty} {c'_{ij}}^2 + \sum_{i=n}^{\infty} \sum_{j=0}^{m-1} {c'_{ij}}^2 + \sum_{i=0}^{n-1} \sum_{j=m}^{\infty} {c'_{ij}}^2, \end{aligned}$$

where, ${c'_{ij}}^2 = \frac{c_{ij}(b-a)^2}{2^{i+j}}$ and

$$c_{ij} = \int_a^b \left(\int_a^b f(x,y) h_i(y) dy \right) h_j(x) dx.$$
$$= \int_a^b \left(\int_{a+k\left(\frac{b-a}{2^i}\right)}^{a+\left(k+\frac{1}{2}\right)\left(\frac{b-a}{2^i}\right)} f(x,y) dy - \int_{a+\left(k+\frac{1}{2}\right)\left(\frac{b-a}{2^i}\right)}^{a+(k+1)\left(\frac{b-a}{2^i}\right)} f(x,y) dy \right) h_j(x) dx$$

Using the mean value theorem of integral calculus we have,

$$a + k\frac{(b-a)}{2^i} \le y_1 \le a + \left(k + \frac{1}{2}\right) \frac{(b-a)}{2^i},$$
$$a + \left(k + \frac{1}{2}\right) \frac{(b-a)}{2^i} \le y_2 \le a + (k+1) \frac{(b-a)}{2^i}.$$

Hence, we obtain

$$c_{ij} = (b-a) \int_a^b \left(f(x,y_1) 2^{-i-1} - f(x,y_2) 2^{-k-1} \right) h_j(x) dx.$$

Again by using the mean value theorem,

$$c_{ij} = 2^{-i-1}(b-a) \int_a^b (f(x,y_1) - f(x,y_2)) h_j(x) dx \quad .$$

Using Lagrange's mean value theorem,

$$= 2^{-i-1}(b-a) \int_a^b \left((y_1 - y_2) \frac{\partial f(x,y^*)}{\partial y} \right) h_j(x) dx \quad \text{where } y_1 \le y^* \le y_2$$
$$= 2^{-i-1}(b-a)(y_1 - y_2) \left(\int_{a+k\left(\frac{b-a}{2^j}\right)}^{a+\left(k+\frac{1}{2}\right)\left(\frac{b-a}{2^j}\right)} \frac{\partial f(x,y^*)}{\partial y} dx \right.$$
$$\left. - \int_{a+\left(k+\frac{1}{2}\right)\left(\frac{b-a}{2^j}\right)}^{a+(k+1)\left(\frac{b-a}{2^j}\right)} \frac{\partial f(x,y^*)}{\partial y} dx \right)$$

$$= 2^{-i-1}(b-a)(y_1-y_2)\left(2^{-j-1}(b-a)\frac{\partial f}{\partial y}(x_1,y^*) - 2^{-j-1}(b-a)\frac{\partial f}{\partial y}(x_2,y^*)\right)$$

Now, we use the mean value theorem of integral calculus

$$\begin{aligned} a + k\frac{(b-a)}{2^j} &\le x_1 \le a + \left(k+\frac{1}{2}\right)\frac{(b-a)}{2^j}, \\ a + \left(k+\frac{1}{2}\right)\frac{(b-a)}{2^j} &\le x_2 \le a + (k+1)\,\frac{(b-a)}{2^j} \\ &\le 2^{-i-j-2}(b-a)^2(y_1-y_2)(x_1-x_2)\frac{\partial^2 f(x^*,y^*)}{\partial x \partial y}. \end{aligned}$$

But for $x_1 \le x* \le x_2,\ (y_1-y_2) \le (b-a)$ and $(x_1-x_2) \le (b-a)$,
We obtain,

$$c_{ij} \le \frac{(b-a)^4}{2^{i+j+2}}M \ \text{ if } \left|\frac{\partial^2 f(x^*,y^*)}{\partial x \partial y}\right| \le M$$

Therefore, ${c'_{ij}}^2 = c_{ij}^2\frac{(b-a)^2}{2^{i+j}} \le \frac{(b-a)^{10}}{2^{3i+3j+4}}M^2$

$$\begin{aligned} \sum_{n=k}^{\infty}\sum_{m=l}^{\infty}{c'_{nm}}^2 &\le \sum_{n=2^{\alpha+1}}^{\infty}\ \sum_{m=2^{\beta+1}}^{\infty}\frac{(b-a)^{10}}{2^{3i+3j+4}}M^2, \quad \alpha,\beta = 0,1,2,\ldots \\ &\le (b-a)^{10}M^2\sum_{n=2^{\alpha+1}}^{\infty}\ \sum_{i=\beta+1}^{\infty}\ \sum_{m=2^i}^{2^{i+1}-1}2^{-3i-3j-4} \\ &\le (b-a)^{10}M^2\sum_{n=2^{\alpha+1}}^{\infty}2^{-3j-4}\sum_{i=\beta+1}^{\infty}(2^{i+1}-1-2^i+1)2^{-3i} \\ &\le (b-a)^{10}M^2\sum_{n=2^{\alpha+1}}^{\infty}2^{-3j-4}\sum_{i=\beta+1}^{\infty}2^{-2i} \\ &\le (b-a)^{10}M^2\sum_{n=2^{\alpha+1}}^{\infty}2^{-3j-4}2^{-2(\beta+1)}\frac{1}{\left(1-\frac{1}{2^2}\right)} \\ &\le \frac{4(b-a)^{10}}{3l^2}M^2 2^{-4}\sum_{j=\alpha+1}^{\infty}\ \sum_{n=2^j}^{2^{j+1}-1}2^{-3j} \\ &\le \frac{4(b-a)^{10}}{3l^2}M^2 2^{-4}\sum_{j=\alpha+1}^{\infty}2^{-2j} \\ &\le \frac{4(b-a)^{10}}{3l^2}M^2 2^{-4}\left(\frac{4}{3}\right)2^{-2(\alpha+1)} \end{aligned}$$

$$
\begin{aligned}
&\le \left(\frac{16}{9}\right)\frac{(b-a)^{10}}{l^2k^2}M^2 2^{-4}\\
&= \left(\frac{16}{144}\right)\frac{(b-a)^{10}}{l^2k^2}M^2
\end{aligned}
$$

Next,

$$
\begin{aligned}
\sum_{n=k}^{\infty}\sum_{m=0}^{l-1} {c'_{nm}}^2 &\le \sum_{n=k}^{\infty}\sum_{m=0}^{l-1}\frac{(b-a)^{10}M^2}{2^{3i+3j+4}} \le \sum_{n=2^{\alpha+1}}^{\infty}\frac{(b-a)^{10}M^2}{2^{3j+4}}\sum_{i=0}^{\beta}\sum_{m=2^i-1}^{2^{i+1}-1}2^{-3i}\\
&\le \sum_{n=2^{\alpha+1}}^{\infty}\frac{(b-a)^{10}M^2}{2^{3j+4}}\sum_{i=0}^{\beta}(2^{-2i}+2^{-3i})\\
&\le \left(\frac{52}{21}\right)2^{-4}(b-a)^{10}M^2\sum_{j=\alpha+1}^{\infty}\sum_{n=2^j}^{2^{j+1}-1}2^{-3j}\\
&\le \left(\frac{52}{336}\right)(b-a)^{10}M^2\sum_{j=\alpha+1}^{\infty}2^{-2j}\\
&\le \left(\frac{52}{336}\right)(b-a)^{10}M^2\left(\frac{2^{-2(\alpha+1)}}{(1-\frac{1}{2^2})}\right)\\
&= \frac{52(b-a)^{10}M^2}{252k^2}.
\end{aligned}
$$

Similarly, we have

$$
\sum_{n=0}^{k-1}\sum_{m=l}^{\infty} c'_{nm}2 \le \frac{52(b-a)^{10}M^2}{252l^2}.
$$

Then

$$
\begin{aligned}
\sum_{n=k}^{\infty}\sum_{m=l}^{\infty} {c'_{nm}}^2 + \sum_{n=k}^{\infty}\sum_{m=0}^{l-1} {c'_{nm}}^2 + \sum_{n=0}^{k-1}\sum_{m=l}^{\infty} {c'_{nm}}^2 &\le \left(\frac{16}{144}\right)\frac{(b-a)^{10}}{l^2k^2}M^2\\
&+\frac{52(b-a)^{10}M^2}{252k^2}+\frac{52(b-a)^{10}M^2}{252l^2}.
\end{aligned}
$$

Hence, we obtain $\|f(x,y)-f_{kl}(x,y)\| \le \frac{(b-a)^{10}M^2}{3}\left(\frac{1}{3l^2k^2}+\frac{13}{21k^2}+\frac{13}{21l^2}\right)$

As $l\to\infty$ and $k\to\infty$ we can get $\|f(x,y)-f_{kl}(x,y)\|\to 0$.

Table 1 Comparison of present method solution with other numerical methods for classical order time fractional Fokker–Planck equation (6.1) at various points of x and t for $\alpha = 1$

t	x	u_{ADM} [8]	u_{VIM} [8]	u_{Exact} given in Eq. (6.3)	u_{Haar}
0.2	0.25	0.076333	0.076333	0.076338	0.0756165
	0.5	0.305333	0.305333	0.305351	0.304392
	0.75	0.687000	0.687000	0.687039	0.686321
	1.0	1.221333	1.221333	1.221403	1.2214
0.4	0.25	0.093167	0.093167	0.093239	0.0958469
	0.5	0.372667	0.372667	0.372956	0.376435
	0.75	0.838500	0.838500	0.839151	0.841761
	1.0	1.490667	1.490667	1.491825	1.49182
0.6	0.25	0.113500	0.113500	0.113882	0.110663
	0.5	0.454000	0.454000	0.455530	0.451238
	0.75	1.021500	1.021500	1.024942	1.02172
	1.0	1.816000	1.816000	1.822119	1.82212

Table 2 Comparison of present method solution with other numerical methods for time fractional Fokker–Planck equation (6.1) at various points of x and t taking $\alpha = 0.5$ and 0.75

t	x	$\alpha = 0.5$			$\alpha = 0.75$		
		u_{ADM} [8]	u_{VIM} [8]	u_{Haar}	u_{ADM} [8]	u_{VIM} [8]	u_{Haar}
0.2	0.25	0.110744	0.091795	0.0900792	0.087699	0.084593	0.0714745
	0.5	0.442978	0.367179	0.421013	0.350796	0.338372	0.329117
	0.75	0.996699	0.826154	0.990531	0.789291	0.761337	0.773339
	1.0	1.771910	1.468717	1.79902	1.403180	1.353488	1.40468
0.4	0.25	0.143997	0.118678	0.13581	0.111718	0.106178	0.0973803
	0.5	0.575909	0.474712	0.587481	0.446872	0.424712	0.431178
	0.75	1.295980	1.068102	1.35217	1.005460	0.955602	0.998822
	1.0	2.303960	1.898849	2.43004	1.787490	1.698849	1.80046
0.6	0.25	0.176478	0.146209	0.167654	0.138479	0.129926	0.116878
	0.5	0.705914	0.584835	0.749162	0.553918	0.519702	0.534521
	0.75	1.588310	1.315878	1.742	1.246320	1.169330	1.24986
	1.0	2.823650	2.339338	3.14621	2.215670	2.078809	2.26291

9 Numerical results and discussion

The following Table 1 shows the comparison of exact solutions with the approximate solutions of different numerical methods for time-fractional Fokker–Planck equation. Agreement between present numerical results with other approximate solutions and exact solutions appears very satisfactory through illustrations in Tables 1 and 2. Table 2 shows the comparison of approximate solutions of fractional order time-fractional Fokker–Planck equation obtained by using two dimensional Haar wavelet

Table 3 Comparison of approximate solutions obtained by using VIM, ADM and Haar wavelet method for time- and space-fractional Fokker–Planck equation (7.1) at various points of x and t taking $\alpha = 1, \beta = 1$

t	x	u_{ADM} [8]	u_{VIM} [8]	u_{Exact} given in Eq. (7.3)	u_{Haar}
0.2	0.25	0.069062	0.069062	0.069073	0.0689468
	0.5	0.276259	0.276250	0.276293	0.274611
	0.75	0.621563	0.621563	0.621659	0.619337
0.4	0.25	0.076250	0.076250	0.076338	0.0753937
	0.5	0.305000	0.305000	0.305351	0.299222
	0.75	0.686250	0.686250	0.687039	0.676175
0.6	0.25	0.084062	0.084063	0.084366	0.0818405
	0.5	0.336250	0.336250	0.337465	0.323833
	0.75	0.756562	0.756562	0.759296	0.733012

Table 4 Comparison of approximate solutions of fractional order time- and space-fractional Fokker–Planck equation (7.1) obtained by using VIM, ADM, OTM and Haar wavelet method at various points of x and t taking $\alpha = \beta = 0.5$ and $\alpha = \beta = 0.75$.

t	x	$\alpha = 0.5$ and $\beta = 0.5$				$\alpha = 0.75$ and $\beta = 0.75$			
		u_{ADM} [8]	u_{VIM} [8]	u_{OTM} [9]	u_{Haar}	u_{ADM} [8]	u_{VIM} [8]	u_{OTM} [9]	u_{Haar}
0.2	0.25	0.060440	0.06111	0.061929	0.0601168	0.063002	0.062922	0.062920	0.0633685
	0.5	0.244329	0.24618	0.248365	0.244247	0.258161	0.256856	0.256782	0.256326
	0.75	0.559866	0.56056	0.562348	0.559936	0.592855	0.587790	0.588104	0.595415
0.4	0.25	0.059620	0.05996	0.061392	0.0591215	0.063371	0.063291	0.063305	0.063968
	0.5	0.242066	0.24303	0.246833	0.241821	0.264157	0.262868	0.262916	0.260722
	0.75	0.558992	0.55902	0.562276	0.558771	0.615589	0.610213	0.611786	0.618446
0.6	0.25	0.059004	0.05898	0.060883	0.0583544	0.063713	0.063642	0.063669	0.0644986
	0.5	0.240363	0.24033	0.245395	0.239941	0.269702	0.268564	0.268707	0.264632
	0.75	0.558407	0.55777	0.562273	0.557834	0.636878	0.631709	0.634637	0.639038

method with the solutions of Adomian decomposition method (ADM) and Variational iteration method (VIM) presented in Ref. [8].

Similarly Tables 3 and 4 show the comparison of approximate solutions obtained by different numerical methods for time- and space-fractional Fokker–Planck equation. It is found that the solutions obtain by using present method are in good agreement with the results presented in Ref. [8] and even better than the results obtained by Operational Tau method (OTM) presented in Ref. [9]. However, the errors may be reduced significantly if we increase level of resolution which prompts more number of collocation points.

10 Conclusion

In this paper, the time and space fractional Fokker–Planck equations have been solved by using two dimensional Haar wavelet method. The obtained results are then compared with exact solutions as well as results obtained by Adomian decomposition method (ADM), Variational iteration method (VIM) and Operational Tau method (OTM) which are available in open literature. These results have been cited in the tables in order to justify the accuracy and efficiency of the proposed scheme. The Haar wavelet technique provides quite satisfactory results in comparison to results obtained by ADM, VIM and OTM [8,9] for the fractional order Fokker–Planck equations as demonstrated in Tables 1, 2, 3 and 4. The main advantage of this Haar wavelet method is that it transfers the whole scheme into a system of algebraic equations for which the computation is easy and simple. In addition, other pretty features of this scheme are its simplicity, applicability and less computational effort. Moreover, the errors may be reduced significantly if we increase level of resolution which prompts more number of collocation points.

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